Morrey spaces–Applications to PDE

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Outline

We overview the properties of Morrey spaces.

Let $0 < q \le p \le \infty$. For an $L^q_{\rm loc}(\mathbb{R}^n)$-function f, its (classical) Morrey norm is defined by

$$\|f\|_{\mathcal{M}^p_q} \equiv \sup_{(x,r) \in \mathbb{R}^{n+1}_+} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,r)} |f(y)|^q {\rm d}y \right)^{\frac{1}{q}}. \qquad (1)$$

The (classical) Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all $f \in L^q_{\rm loc}(\mathbb{R}^n)$ for which the norm $\|f\|_{\mathcal{M}^p_q}$ is finite.

Example

Let $0 < q \leq p < \infty$. Let B be an open ball. Then

$$\|\chi_B\|_{\mathcal{M}^p_q} = \|\chi_B\|_{L^p}. \tag{2}$$

In fact, it is easy to see that

$$\|\chi_B\|_{\mathcal{M}^p_q} \leq \|\chi_B\|_{L^p} \tag{3}$$

from Hölder's inequality.

If we write out the norm $\|\chi_B\|_{\mathcal{M}^p_q}$ in full, then

$$\|\chi_B\|_{\mathcal{M}^p_q} = \sup_{(x,r)\in\mathbb{R}^{n+1}_+} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,r)} |\chi_B(y)|^q \mathrm{d}y\right)^{\frac{1}{q}}.$$

We can calculate and evaluate the integral precisely. The result is:

$$\|\chi_B\|_{\mathcal{M}^p_q} = \sup_{(x,r)\in\mathbb{R}^{n+1}_+} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} |B(x,r)\cap B|^{\frac{1}{q}} \geq |B|^{\frac{1}{p}} = \|\chi_B\|_{L^p}. \tag{4}$$

Combining (3) and (4), we obtain (2).

A major problem concerning simple/fundamental function spaces $\mathrm{BC}(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ is that neither of them contains $|\cdot|^{-\alpha}$ for any $\alpha \in \mathbb{R}$. However, this simplest function appears everywhere but to handle this function, we need to restrict its domain. The Morrey space $\mathcal{M}^{\frac{n}{\alpha}}_q(\mathbb{R}^n)$ with $1 < q < \frac{n}{\alpha}$ contains $|\cdot|^{-\alpha}$. Here and below, $B(r)$ abbreviates $B(x, r)$ with x the origin.

The Morrey space $\mathcal{M}^{\frac{n}{\alpha}}_q(\mathbb{R}^n)$ with $1 < q < \frac{n}{\alpha}$ contains $f(x) = f_\alpha(x) \equiv |x|^{-\alpha}$, $x \in \mathbb{R}^n$. To check this, we observe that

$$\sup_{x \in \mathbb{R}^n} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(B(x,r))} = |B(r)|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(B(r))}. \qquad (5)$$

Example

Let $0 < q < p < \infty$, and let $\alpha, \beta \in \mathbb{R}$.

(1) Let $f(x) = f_\alpha(x) \equiv |x|^\alpha$, $x \in \mathbb{R}^n$. Then $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ if and only if $\alpha = -\dfrac{n}{p}$.

(2) Let $g \equiv f\chi_{B(1)}$. Then $g \in \mathcal{M}^p_q(\mathbb{R}^n)$ if and only if $\alpha \geq -\dfrac{n}{p}$. Likewise let $h \equiv f\chi_{\mathbb{R}^n \setminus B(1)}$. Then $h \in \mathcal{M}^p_q(\mathbb{R}^n)$ if and only if $\alpha \leq -\dfrac{n}{p}$.

(3) Let $k \equiv f_\alpha\chi_{B(1)} + f_\beta\chi_{\mathbb{R}^n \setminus B(1)}$. From (2), we see that $k \in \mathcal{M}^p_q(\mathbb{R}^n)$ if and only if $\alpha \geq -\dfrac{n}{p} \geq \beta$.

Example (Toothbrush)

Let $0 < q < p < \infty$. We choose α so that $\left(\frac{\alpha}{N}\right)^{\frac{1}{p}} = \alpha^{\frac{1}{q}}$. We divide equally $[0,1]^n$ into N^n cubes to have $Q_1^N, Q_2^N, \ldots, Q_{N^n}^N$. We will consider $f = f_N \equiv \sum_{j=1}^{N^n} \chi_{\alpha Q_j^N} \in L_c^\infty(\mathbb{R}^n)$. We show that $\|f\|_{\mathcal{M}_q^p} \sim \|f\|_{L^q}$.

To this end we use

$$\|f\|_{\mathcal{M}_q^p} \equiv \|f\|_{\mathcal{M}_q^p}^{\mathcal{Q}} = \sup_{(x,r)\in\mathbb{R}_+^{n+1}} |Q(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q(x,r)} |f(y)|^q \mathrm{d}y\right)^{\frac{1}{q}}.$$

Since f is supported on $[0,1]^n$, we may assume that $Q(x,r)$ runs over all cubes contained in $[0,1]^n$. We first note that the supremum is attained by letting $r=2\alpha$: If we choose $x \in \mathbb{R}^n$ suitably, then we have $|Q(x,r)|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^q(Q(x,r))} \lesssim |Q(r)|^{\frac{1}{p}} = \|\chi_{\alpha Q_j}\|_{L^p} = \|f\|_{L^q}$. If $2r<\alpha$ instead, then we have $|Q(x,r)|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^q(Q(x,r))} \le |Q(x,r)|^{\frac{1}{p}} < \|f\|_{L^q}$ no matter where x is. Thus, to show the result, we have only to consider the cubes $Q(x,r)$ with $\alpha \le 2r \le 1$. If $Q(x,r)$ intersects Q_j^N and Q_k^N with $1 \le j < k \le N^n$, then there exists $R \in \mathcal{Q}$ such that $R \cap \operatorname{supp}(f)$ is realized as the union $\{Q_j^N\}_{j \in J}$ for some $J \subset \{1,2,\ldots,N^n\}$, that R contains $Q(x,r)$ and that $|R| \lesssim_\alpha |Q(x,r)|$. Hence, by translating R if necessary, we may assume $R=[0,kN^{-1}]^n$ for some $k=1,2,\ldots,N$. For this R,

$|Q(x,r)|^{\frac{1}{p}-\frac{1}{q}}\|f\| \ldots \le \|f\| \ldots$

Let $0 < \alpha < n$. Let I_α be the fractional integral operator given by

$$I_\alpha f(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \mathrm{d}y \quad (x \in \mathbb{R}^n) \tag{6}$$

for a non-negative measurable function f.

Theorem

Let $1 < p \le p_0 < \infty$ $1 < q \le q_0 < \infty$ *and* $1 < r \le r_0 < \infty$. *Assume*

$$r < q, n/q_0 \le \alpha < n/p_0,$$

$$1/r_0 = 1/p_0 + 1/q_0 - \alpha/n, \quad r/r_0 = p/p_0.$$

Then

$$\|g I_\alpha f\|_{\mathcal{M}^{r_0}_r} \le C \|f\|_{\mathcal{M}^{p_0}_p} \|g\|_{\mathcal{M}^{q_0}_q}.$$

Here is another application.

$$L_0 := -\mathrm{div} A\nabla = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right).$$

We consider an elliptic differential operator L with non-smooth coefficients generated by a symmetric matrix $A = [a_{ij}]_{i,j=1}^{n} \in (L^\infty(\mathbb{R}^n))^{n^2}$ given by

$$L := I + L_0 = I - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right).$$

For an $L^q_{\rm loc}(\mathbb{R}^n)$-function f, its non-local Morrey norm is defined by

$$\|f\|_{M^p_q} \equiv \sup_{(x,r)\in\mathbb{R}^{n+1}_+, r\ge 1} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,r)} |f(y)|^q {\rm d}y\right)^{\frac{1}{q}}. \quad (7)$$

The non-local Morrey space $M^p_q(\mathbb{R}^n)$ is the set of all $f \in L^q_{\rm loc}(\mathbb{R}^n)$ for which the norm $\|f\|_{\mathcal{M}^p_q}$ is finite.

Lemma

Let $p \geq 2$. *Then for all* $u \in W^1 M^p_2(\mathbb{R}^n)$

$$\|u\|_{M^p_2} + \|Du\|_{(M^p_2)^n} \prec \|(1-\Delta)^{-1/2} Lu\|_{M^p_2}.$$

We will show the well-posedness for the Cauchy problem of a two dimensional semi-linear elliptic parabolic system

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) + \mathrm{div}(u(t,x)\nabla\psi(t,x)) = 0, & (t,x) \in \mathbb{R}^3_+ \\ -\Delta\psi(t,x) = \kappa u(t,x), & (t,x) \in \mathbb{R}^3_+ \\ u(0,x) = u_0(x), & x \in \mathbb{R}^2 \end{cases} \tag{8}$$

with $\kappa = \pm 1$. When $\kappa = 1$, system (8) describes a model for the chemotaxis and the system is called the (simplified) Keller–Segel equation, the Jäger–Luckhaus system or the Nagai model. When $\kappa = -1$, system (8) is called as a mono-polar drift-diffusion system for the semi-conductor simulation. In this talk, we will handle both of them but we suppose $\kappa = 1$ to simpify.

We consider the integral version of the Keller–Segel equation.

$$u = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathrm{div}[u(s)\nabla(-\Delta)^{-1}u(s)]\mathrm{d}s \quad (t > 0). \quad (9)$$

We present a typical result.

Theorem

Let $\kappa = \pm 1$. Let $1 \le q \le p < 2$. Define $\delta \equiv 2 - \dfrac{2}{p}$. Write $I = [0, T)$. Then for $u_0 \in \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2)$, there exists $T > 0$ and a unique solution

$$u \in C(I; \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2)) \cap L^2(I; \dot{\mathcal{N}}^{1-\delta}_{pq2}(\mathbb{R}^2)) \cap L^2(I; \dot{B}^{-1}_{\infty 1}(\mathbb{R}^2))$$

to (9)*. Besides u satisfies*

$$u \in C(\mathrm{Int}(I); \dot{\mathcal{N}}^{2-\delta}_{pq2}(\mathbb{R}^2)) \cap C^1(\mathrm{Int}(I); \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2))$$

and the flow map $u_0 \in \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2) \mapsto u \in C(I; \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2))$ is Lipschitz continuous.

Tools to deal with this PDE

- paraproduct
- maximal regularity
- fixed point theorem

Why do I want to use Morrey spaces?

1. Generic Morrey spaces fail to be
   - separable,
   - reflexive.
2. $L^p(\mathbb{R}^n) = \mathcal{M}^p_p(\mathbb{R}^n) \subset \mathcal{M}^p_q(\mathbb{R}^n)$. This in turn implies that we are interested in the generic case of $p \neq q$.
3. Morrey spaces do not have $C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ as a dense subspace. Nor do they have $L^p(\mathbb{R}^n)$ as a dense subspace.
4. Morrey spaces sometimes nicely extend the Sobolev embedding.
5. But sometimes Morrey spaces do not give us any nice extension of the Sobolev embedding. (Embeddings between Besov–Morrey spaces and Triebel–Lizorkin–Morrey spaces).

Outline

Polynomials

Denote by $\mathcal{P} = \mathcal{P}(\mathbb{R}^n)$ the set of all polynomials. Then using the standard mapping $f \mapsto F_f$ we can regard $\mathcal{P}(\mathbb{R}^n)$ as the subset of $\mathcal{S}'(\mathbb{R}^n)$. Denote by $\mathcal{P}_d(\mathbb{R}^n)$ the set of all polynomial functions with degree less than or equal to d, so that $\mathcal{P}(\mathbb{R}^n) \equiv \bigcup_{d=0}^{\infty} \mathcal{P}_d(\mathbb{R}^n)$. It is understood that $\mathcal{P}_{-1}(\mathbb{R}^n) = \{0\}$.

Littlewood–Paley patch

Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\chi_{B(2)} \le \psi \le \chi_{B(4)}$. Write $\varphi = \psi - \psi(2\cdot)$. For $f \in \mathcal{S}'(\mathbb{R}^n)$, we define

$$\varphi_j(D)f = \mathcal{F}^{-1}[\varphi(2^{-j}\cdot)\mathcal{F}f].$$

This definition makes sense for $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$.

For $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$, we define

$$\|f\|_{\dot{\mathcal{N}}^s_{pqr}} \equiv \left(\sum_{j=-\infty}^{\infty} (2^{js} \|\varphi_j(D)f\|_{\mathcal{M}^p_q})^r \right)^{\frac{1}{r}}, \qquad (10)$$

$$\|f\|_{\dot{\mathcal{E}}^s_{pqr}} \equiv \left\| \left(\sum_{j=-\infty}^{\infty} 2^{jrs} |\varphi_j(D)f|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}^p_q}. \qquad (11)$$

The spaces $\dot{\mathcal{N}}^s_{pqr}(\mathbb{R}^n)$, which we call the *homogeneous Besov–Morrey space* and the *homogeneous Triebel–Lizorkin–Morrey space* respectively, and $\dot{\mathcal{E}}^s_{pqr}(\mathbb{R}^n)$ are the sets of all $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ for which the norms $\|f\|_{\dot{\mathcal{N}}^s_{pqr}}$ and $\|f\|_{\dot{\mathcal{E}}^s_{pqr}}$ are finite, respectively.

We recall the definition of $\mathcal{S}_\infty(\mathbb{R}^n)$ and $\mathcal{S}'_\infty(\mathbb{R}^n)$.

Definition ($\mathcal{S}_\infty(\mathbb{R}^n)$ and $\mathcal{S}'_\infty(\mathbb{R}^n)$)

Define the *Lizorkin function space* $\mathcal{S}_\infty(\mathbb{R}^n)$ by $\mathcal{S}_\infty(\mathbb{R}^n) \equiv \mathcal{S}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)^\perp$. Equip $\mathcal{S}_\infty(\mathbb{R}^n)$ with the topology induced by $\mathcal{S}(\mathbb{R}^n)$. The *Lizorkin distribution space* $\mathcal{S}'_\infty(\mathbb{R}^n)$ is the topological dual space $\mathcal{S}_\infty(\mathbb{R}^n)$. That is, define

$$\mathcal{S}'_\infty(\mathbb{R}^n) = \{F : \mathcal{S}_\infty(\mathbb{R}^n) \to \mathbb{C} \,:\, F \text{ is continuous and } \mathbb{C}\text{-linear}\}.$$

As for the space $\dot{\mathcal{N}}^s_{pqr}(\mathbb{R}^n)$, we have a simpler representation.

Theorem

Let $1 \le q \le p < \infty$, $1 \le r \le \infty$ *and* $s \in \mathbb{R}$. *Let* $f \in \dot{\mathcal{N}}^s_{pqr}(\mathbb{R}^n)$. *Then there exists a sequence* $\{P_j\}_{j=1}^\infty$ *of polynomials of degree* $d = \max\left(-1, \left[s - \frac{n}{p}\right]\right)$ *such that*

$$g = \lim_{j\to\infty}\left(P_j + \sum_{k=-j}^{\infty} \varphi_j(D)f\right)$$

exists in $\mathcal{S}'(\mathbb{R}^n)$. *In particular,* $f = \lim_{j\to\infty}\left(P + P_j + \sum_{k=-j}^{\infty} \varphi_j(D)f\right)$ *holds in the topology of* $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}_d(\mathbb{R}^n)$ *for some* $P \in \mathcal{P}(\mathbb{R}^n)$.

Theorem (Paraproduct)

Suppose that we have parameters $p_1, p_2, p, q_1, q_2, q, r, s_1, s_2$ *satisfying*

$$1 \le q_1 \le p_1 < \infty, \quad 1 \le q_2 \le p_2 < \infty,$$

$$1 \le q \le p < \infty, \quad 1 \le r < \infty, \quad s_1, s_2 > 0,$$

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then $\|f \cdot g\|_{\dot{\mathcal{N}}^{s_1}_{pqr}} \lesssim \|f\|_{\dot{\mathcal{N}}^{s_2}_{p_1 q_1 \infty}} \|g\|_{\dot{\mathcal{N}}^{s_1}_{p_2 q_2 r}} + \|f\|_{\dot{\mathcal{N}}^{s_1}_{pqr}} \|g\|_{L^\infty}$ *for all* $f \in \dot{\mathcal{N}}^{s_2}_{p_1 q_1 \infty}(\mathbb{R}^n) \cap \dot{\mathcal{N}}^{s_1}_{pqr}(\mathbb{R}^n)$ *and* $g \in \dot{\mathcal{N}}^{s}_{p_2 q_2 r}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Outline

With this in mind, let us recall the local means for $\dot{\mathcal{N}}^s_{pqr}(\mathbb{R}^n)$.

Theorem (S.-Tanaka/Yuan and Yang/Rosenthal)

Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\chi_{B(1)} \le \psi \le \chi_{B(2)}$. Also let $1 \le q \le p < \infty$, $1 \le r \le \infty$ and $s \in \mathbb{R}$. Then for $f \in \dot{\mathcal{N}}^s_{pqr}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}_d(\mathbb{R}^n)$ with $d = \max\left(-1, \left[s - \frac{n}{p}\right]\right)$,

$$\|f\|_{\dot{\mathcal{N}}^s_{pqr}} \sim \sum_{k=1}^{n} \left(\sum_{j=-\infty}^{\infty} (2^{js} \|2^{jn}(\partial_k^{d+1}\psi)(2^j \cdot) * f\|_{\mathcal{M}^p_q})^r \right)^{\frac{1}{r}}.$$

This is one of the simplest form using the functions employed so far.

Here are variants. First, let us stop using the convolution with ψ and let us use an expression that is connected more directly with PDE.

Theorem (Liang, S., Ullrich, Yang and Yuan)

Let $1 \le q \le p < \infty$, $1 \le r \le \infty$ *and* $s \in \mathbb{R}$. *Assume that* $d = \max\left(-1, \left[s - \frac{n}{p}\right]\right) \in \{0, -1\}$. *That is* $s - \frac{n}{p} < 1$. *Then for* $f \in \dot{\mathcal{N}}^s_{pqr}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)/\mathbb{C}$,

$$\|f\|_{\dot{\mathcal{N}}^s_{pqr}} \sim \sum_{k=1}^{n} \left(\sum_{j=-\infty}^{\infty} (2^{j(s-1)} \|\partial_k [e^{4^{-j}\Delta} f]\|_{\mathcal{M}^p_q})^r \right)^{\frac{1}{r}}.$$

A passage to the continuous variable t from the discrete variable j can be done with ease.

Theorem (Liang, S., Ullrich, Yang and Yuan)

Let $1 \le q \le p < \infty$, $1 \le r \le \infty$ and $s \in \mathbb{R}$. Assume that $d \in \{0, -1\}$. Then for $f \in \dot{\mathcal{N}}^s_{pqr}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)/\mathbb{C}$,

$$\|f\|_{\dot{\mathcal{N}}^s_{pqr}} \sim \sum_{k=1}^{n} \left(\int_0^\infty \sum_{j=-\infty}^{\infty} \left(t^{-\frac{s-1}{2}} \|\partial_k[e^{t\Delta} f]\|_{\mathcal{M}^p_q}\right)^r \frac{dt}{t} \right)^{\frac{1}{r}}.$$

Outline

One important observation is that the Morrey norm $\|\cdot\|_{\mathcal{M}^p_q}$ and the Besov–Morrey norm $\|\cdot\|_{\dot{\mathcal{N}}^S_{pqu}}$ are not that different in front of $\partial_k e^{t\Delta}$:

Theorem (S. and Nogayama)

Let $1 \le q \le p < \infty$, $1 \le r \le \infty$ and $s, S \in \mathbb{R}$. Assume that $d \in \{0, -1\}$. Then for $f \in \dot{\mathcal{N}}^s_{pqr}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)/\mathbb{C}$,

$$\|f\|_{\dot{\mathcal{N}}^s_{pqr}} \sim \sum_{k=1}^{n} \left(\int_0^\infty \sum_{j=-\infty}^{\infty} (t^{-\frac{1}{2}S+\frac{1}{2}s} \|\sqrt{t}\partial_k[e^{t\Delta} f]\|_{\dot{\mathcal{N}}^S_{pqu}})^r \frac{dt}{t} \right)^{\frac{1}{r}}.$$

It is important that we have freedom in the choice of S and u. Especially, the choice of u is important because the paraproduct estimates are somewhat weak in that we need to use $\|\cdot\|_{\mathcal{N}^{\frac{n}{p}}_{pq1}}$. It matters that we are forced to take 1 in the third index. The space $\mathcal{N}^{\frac{n}{p}}_{pq1}(\mathbb{R}^n)$ can be embedded into $L^\infty(\mathbb{R}^n)$ mainly by the use of the triangle inequality and the Plancherel–Polya–Nikolski'i inequality.

Outline

We extend the result by Ogawa and Shimizu in 2010 to Besov–Morrey spaces. We start with the heat equation.

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \mathbb{R}^{n+1}_+, \\ u(0,\cdot) = u_0 & \text{on } \mathbb{R}^n. \end{cases} \tag{12}$$

Theorem

Let $1 \le q \le p < \infty$, $1 \le \rho \le \infty$. *Consider the heat equation* (12) *with* $f \in L^\rho([0,\infty);\dot{\mathcal{N}}^0_{pq\rho}(\mathbb{R}^n))$ *and* $u_0 \in \dot{\mathcal{N}}^{2-2/\rho}_{pq\rho}(\mathbb{R}^n)$. *Then*

$$\|\partial_t u\|_{L^\rho([0,\infty);\dot{\mathcal{N}}^0_{pq\rho})} + \|\nabla^2 u\|_{L^\rho([0,\infty);\dot{\mathcal{N}}^0_{pq\rho})} \lesssim \|u_0\|_{\dot{\mathcal{N}}^{2-2/\rho}_{pq\rho}} + \|f\|_{L^\rho([0,\infty);\dot{\mathcal{N}}^0_{pq\rho})}.$$

Theorem

Let $\kappa = \pm 1$. *Let* $1 \le q \le p < 2$. *Define* $\delta \equiv 2 - \dfrac{2}{p}$. *Write* $I = [0, T)$. *Then for* $u_0 \in \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2)$, *there exists* $T > 0$ *and a unique solution*

$$u \in C(I; \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2)) \cap L^2(I; \dot{\mathcal{N}}^{1-\delta}_{pq2}(\mathbb{R}^2)) \cap L^2(I; \dot{B}^{-1}_{\infty 1}(\mathbb{R}^2))$$

to (9). *Besides u satisfies*

$$u \in C(\mathrm{Int}(I); \dot{\mathcal{N}}^{2-\delta}_{pq2}(\mathbb{R}^2)) \cap C^1(\mathrm{Int}(I); \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2))$$

and the flow map $u_0 \in \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2) \mapsto u \in C(I; \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2))$ *is Lipschitz continuous.*

Other estimates I

(1) Let $1 \le q \le p < \infty$ and $1 \le \rho < \infty$. Then

$$\left(\int_0^\infty (\|\nabla \exp(t\Delta) u_0\|_{\dot{\mathcal{N}}^0_{pq1}})^\rho \mathrm{d}t \right)^{\frac{1}{\rho}} \lesssim \|u_0\|_{\dot{\mathcal{N}}^{1-\frac{2}{\rho}}_{pq\rho}}$$

for all $u_0 \in \dot{\mathcal{N}}^{1-\frac{2}{\rho}}_{pq\rho}(\mathbb{R}^n)$.

(2) Let $1 \le q \le p < \infty$ and $1 \le \rho \le \infty$. Then

$$\sup_{t>0} \|\nabla \exp(t\Delta) u_0\|_{\dot{\mathcal{N}}^0_{pq\rho}} \lesssim \|u_0\|_{\dot{\mathcal{N}}^1_{pq\rho}}$$

for all $u_0 \in \dot{\mathcal{N}}^1_{pq\rho}(\mathbb{R}^n)$.

(3) For all $u_0 \in \dot{B}^0_{\infty 2}(\mathbb{R}^n)$,

$$\left(\int_0^\infty (\|\nabla \exp(t\Delta) u_0\|_{\dot{B}^0_{\infty 1}})^2 \mathrm{d}t \right)^{\frac{1}{2}} \lesssim \|u_0\|_{\dot{B}^0_{\infty 2}}.$$

Other estimates II

Let $1 \le q \le p < \infty$ and $1 \le \rho \le \infty$. Then

$$\left\| \nabla \int_0^t \exp((t-s)\Delta) f(s) \mathrm{d}s \right\|_{L^\rho(0,\infty;\dot{\mathcal{N}}^0_{pq\rho})} \lesssim \|f\|_{L^\rho(0,\infty;\dot{\mathcal{N}}^{-1}_{pq\rho})}$$

for all $f \in L^\rho(0,\infty;\dot{\mathcal{N}}^{-1}_{pq\rho}(\mathbb{R}^n))$.

Outline

We recall the following embedding:

Theorem (S.)

Let $1 \le q < p < \infty$, $1 \le r \le \infty$ *and* $s \in \mathbb{R}$. *Then*

$$\dot{\mathcal{N}}^s_{pq\,\min(q,r)}(\mathbb{R}^n) \hookrightarrow \dot{\mathcal{E}}^s_{pqr}(\mathbb{R}^n) \hookrightarrow \dot{\mathcal{N}}^s_{pq\infty}(\mathbb{R}^n).$$

This is a remarkable contrast to the well-known embedding:

$$\dot{B}^s_{p\,\min(p,r)}(\mathbb{R}^n) = \dot{\mathcal{N}}^s_{pp\,\min(p,r)}(\mathbb{R}^n) \hookrightarrow \dot{B}^s_{pr}(\mathbb{R}^n) = \dot{\mathcal{E}}^s_{ppr}(\mathbb{R}^n)$$

$$\hookrightarrow \dot{B}^s_{p\,\max(p,r)}(\mathbb{R}^n) = \dot{\mathcal{N}}^s_{pp\,\max(p,r)}(\mathbb{R}^n),$$

We also have a similar phenomenon.

Theorem (Haroske, Skrzypczak)

Let $1 \le q < p < \infty$, $1 \le r \le \infty$ *and* $s \in \mathbb{R}$. *Also let* $1 \le q^\dagger < p^\dagger < \infty$ *and* $s^\dagger \in \mathbb{R}$. *Assume*

$$\frac{p}{q} = \frac{p^\dagger}{q^\dagger}, \quad s - \frac{n}{p} = s^\dagger - \frac{n}{p^\dagger}.$$

Then

$$\dot{\mathcal{E}}^s_{pqr}(\mathbb{R}^n) \hookrightarrow \dot{\mathcal{N}}^{s^\dagger}_{p^\dagger q^\dagger \infty}(\mathbb{R}^n)$$

This is a good contrast to the embedding:

$$\dot{B}^s_{ppr}(\mathbb{R}^n) = \dot{\mathcal{E}}^s_{pqr}(\mathbb{R}^n) \hookrightarrow \dot{B}^s_{p^\dagger p^\dagger p}(\mathbb{R}^n) = \dot{\mathcal{N}}^{s^\dagger}_{p^\dagger p^\dagger p}(\mathbb{R}^n).$$

Outline

Theorem (Hakim, Nakamura, S.)

Let $1 \leq q \leq p < \infty$. Then $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ satisfies $e^{t\Delta} f \to f$ in $\mathcal{M}^p_q(\mathbb{R}^n)$ if and only if f can be approximated in the Morrey norm $\| \cdot \|_{\mathcal{M}^p_q}$ by Morrey functions whose derivative up to order 1 belongs to $\mathcal{M}^p_q(\mathbb{R}^n)$.

Theorem (Nogayama, S.)

Let $1 \leq q < p < \infty$, $1 \leq r \leq \infty$ and $s \in \mathbb{R}$. Then any $f \in \dot{\mathcal{N}}^s_{pqr}(\mathbb{R}^n)$ satisfies $e^{t\Delta} f \to f$ in $\dot{\mathcal{N}}^s_{pqr}(\mathbb{R}^n)$ as $t \downarrow 0$.

Outline

Theorem

Let $1 \le q < p < \infty$, $1 \le r \le \infty$ and $s > \frac{1}{p}$. Then $\mathrm{Tr}_{\mathbb{R}^n}$ maps $\dot{\mathcal{N}}^s_{pqr}(\mathbb{R}^n)$ surjectively to $\dot{\mathcal{N}}^{s-\frac{1}{p}}_{pqr}(\mathbb{R}^n)$, while $\mathrm{Tr}_{\mathbb{R}^n}$ maps $\dot{\mathcal{E}}^s_{pqr}(\mathbb{R}^n)$ surjectively to $\dot{\mathcal{E}}^{s-\frac{1}{p}}_{pqq}(\mathbb{R}^n)$. In particular, $\mathrm{Tr}_{\mathbb{R}^n}$ maps $W^s\mathcal{M}^p_q(\mathbb{R}^n)$ (the case of $r = 2$) surjectively to $\dot{\mathcal{E}}^{s-\frac{1}{p}}_{pqq}(\mathbb{R}^n)$.

Around a decade ago, Besov-type spaces and Triebel-Lizorkin-type spaces are defined by Yang and Yuan. It turned out that Besov-type spaces are different from Besov–Morrey spaces but that Triebel-Lizorkin-type spaces are the same as Triebel-Lizorkin–Morrey spaces.

Thank you for your attention!